

WHITNEY STRATIFICATIONS AND THE CONTINUITY OF LOCAL LIPSCHITZ KILLING CURVATURES

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ABSTRACT. We prove that local Lipschitz Killing curvatures of definable sets in a polynomially bounded o-minimal structure are continuous along strata of Whitney stratifications and locally Lipschitz if the stratifications are (w)-regular.

1. INTRODUCTION

The density of a set $A \subset \mathbb{R}^n$ at a point $x \in \mathbb{R}^n$ is the following limit if it exists

$$\Theta_d(A, x) = \lim_{t \rightarrow 0} \frac{\mathcal{H}^d(A \cap \mathbf{B}_{(x,r)}^n)}{\mu_d r^d},$$

where $\mathbf{B}_{(x,r)}^n$ is the ball in \mathbb{R}^n centered at x of radius r , \mathcal{H}^d is the d -dimensional Hausdorff measure and μ_d is the volume of the unit ball of dimension d .

The existence of density of subanalytic sets was proved by K. Kurdyka and G. Raby in [7]. In fact, if A is a smooth manifold then density of A is 1 at every point in A . The notion of density of A , therefore, not interesting at regular points. If A is a stratified set meaning the set A together with its stratification, all singular points of A will be in strata of dimensions less than d . It is natural to ask how the density of A varies along those strata. In 1988, D. Trotman conjectured that the density of subanalytic sets is continuous on the strata of Whitney stratifications (stratifications satisfying Whitney's (b)-regular condition). G. Comte, in his thesis (1998), proved this conjecture for (w)-regular stratifications (see also [2]). In [13], G. Valette gave the positive answer for the conjecture. He also proved that the density is locally Lipschitz if the stratification is (w)-regular.

G. Comte and M. Merle [3] generalized Comte's result to what are called local Lipschitz Killing curvatures, introduced by A. Bernig and L. Bröcker [1]. To be precise, let A be a compact subset of \mathbb{R}^n , for $k \in \{0, \dots, n\}$, the k -th Lipschitz Killing of the set A is defined as follows

$$\Lambda_k(A) := c(n, k) \int_{P \in \mathbb{G}_n^k} \int_{x \in P} \chi(A \cap \pi_P^{-1}(x)) d\mathcal{H}^k(x) dP, \quad (1.1)$$

where χ is the Euler-Poincaré characteristic, dP is the standard probability measure of the Grassmannian \mathbb{G}_n^k , π_P is the orthogonal projection from \mathbb{R}^n onto P and

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$c(n, k) = \Gamma(\frac{n+1}{2})\Gamma(\frac{1}{2})/\Gamma(\frac{n-k+1}{2})\Gamma(\frac{k+1}{2})$ with $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$. Let $x \in A$. If the limit

$$\Lambda_k^{\text{loc}}(A, x) := \lim_{r \rightarrow 0} \frac{1}{\mu_k r^k} \Lambda_k(A \cap \mathbf{B}_{(x,r)}^n) \quad (1.2)$$

exists we call it the k -th **local Lipschitz Killing curvature** of A at x . We would like to notice, as a consequence of the Cauchy-Crofton formula, that $\Lambda_d(A, x) = \Theta_d(A, x)$. In [3], the authors proved that local Lipschitz Killing curvatures of sub-analytic sets exist and are continuous along strata of (w)-regular stratifications.

In this paper we deal with the question whether the result of Comte and Merle still holds for Whitney stratifications. We consider the problem in the framework of o-minimal structures which is considered as a generalization of semialgebraic and subanalytic geometries. By improving the techniques developed in [13] to study the invariance of the density, we prove that local Lipschitz Killing curvatures of a definable sets in a polynomially bounded o-minimal structure are continuous along strata of Whitney stratifications. This result does not hold for o-minimal structure which is not polynomially bounded. An example in [12] show that, in general, the density of a definable set is not continuous along strata of Whitney stratifications. We show furthermore that if the stratifications are (w)-regular, then the local Lipschitz Killing curvature are locally Lipschitz. This fact is true for every o-minimal structure, it is not necessary to assume to be polynomially bounded.

In Section 2, we recall definitions of o-minimal structures and stratification of definable sets. Section 3 presents some preliminary results of definable sets that is a preparation for proving Proposition 4.2 which is the key of the paper. The results of the continuity of local Lipschitz Killing curvatures for Whitney stratifications and for (w)-regular stratifications are presented in Section 4.

2. DEFINABLE STRATIFICATIONS

2.1. O-minimal structures. A **structure** on the real closed field $(\mathbb{R}, +, \cdot)$ is a family $\mathcal{D} = (\mathcal{D}_n)_{n \in \mathbb{N}}$ satisfying the following properties:

- (1) \mathcal{D}_n is a boolean algebra of subsets of \mathbb{R}^n ,
- (2) If $A \in \mathcal{D}_n$ then $\mathbb{R} \times A \in \mathcal{D}_{n+1}$ and $A \times \mathbb{R} \in \mathcal{D}_{n+1}$,
- (3) \mathcal{D}_n contains the zero sets of all polynomials in n variables,
- (4) If $A \in \mathcal{D}_n$ then its image under projection onto the first $n-1$ coordinates in \mathbb{R}^{n-1} is in \mathcal{D}_{n-1} .

A structure \mathcal{D} is said to be **o-minimal** if in addition

- (5) Any set $A \in \mathcal{D}_1$ is a finite union of open intervals and points.

Elements of \mathcal{D}_n for any n are called **\mathcal{D} -sets** (or definable sets) of \mathcal{D} . A map between two \mathcal{D} sets is said to be a **\mathcal{D} -map** (or definable map) if its graph is a \mathcal{D} -set.

A structure \mathcal{D} is said to be **polynomially bounded** if for every \mathcal{D} -function $f : \mathbb{R} \rightarrow \mathbb{R}$, there exist $a > 0$ and $n \in \mathbb{N}$ such that $|f(x)| \leq x^n$ for all $x > a$.

The class of semialgebraic sets, the class of globally subanalytic sets are examples of polynomially bounded o-minimal structures. We refer the reader to [14, 15, 4, 8] as good references for studying more about o-minimal structures. In the paper, we will use the following properties of \mathcal{D} -sets without citations.

- 1) Uniform Bounded on Fibres (see [14], Chapter 3, (2.13))
- 2) Curve Selection (see [14], Chapter 6, (1.5))
- 3) Hardt's triviality theorem (see [14], Chapter 9, (1.7) or [4], Theorem 5.22).

2.2. Stratifications.

Definition 2.1. Let A be a \mathcal{D} -subset of \mathbb{R}^n . Let $p \in \mathbb{N}$. A C^p **\mathcal{D} -stratification** (or a stratification for simplicity) of A is a partition $\mathcal{S} = \{S_\alpha\}_\alpha$ of A into finitely many C^p , connected \mathcal{D} -submanifolds of \mathbb{R}^n , called **strata**, such that the frontier condition is satisfied meaning if $S_\alpha, S_\beta \in \mathcal{S}$, $S_\alpha \neq S_\beta$, $\text{cl}(S_\alpha) \cap S_\beta \neq \emptyset$, then $S_\beta \subset S_\alpha$.

If we denote by S^k the union of strata of dimensions $\leq k$ of the stratification \mathcal{S} , then A can be described as filtration of skeletons

$$A = S^d \supseteq S^{d-1} \supseteq \dots \supseteq S^l \neq \emptyset$$

such that each difference $\dot{S}^k = S^k \setminus S^{k-1}$ is an k -dimensional C^p \mathcal{D} -submanifold of \mathbb{R}^n or empty. Connected components of \dot{S}^k coincide with strata of dimension k of \mathcal{S} .

The set A together with its stratification \mathcal{S} is called **stratified set** and denoted by (A, \mathcal{S}) . A vector field v defined on A is called **stratified vector field** with respect to the stratification \mathcal{S} if $v(x) \in T_x S$, $x \in S \in \mathcal{S}$.

Suppose that (γ) is some regularity condition defined on pairs of submanifolds of \mathbb{R}^n . A stratification of A is said to be (γ) -regular if the condition (γ) is satisfied for every pair of strata of the stratification. In the following, we recall definitions of regularity conditions which we will deal with in next sections.

Let X, Y be C^1 \mathcal{D} -submanifolds of \mathbb{R}^n . Let $z \in \text{cl}(X) \cap Y$.

Whitney condition (b)— for any sequence $\{x_k\}_{k \in \mathbb{N}}$ in X and any sequence $\{y_k\}_{k \in \mathbb{N}}$ in Y , each converging to z such that the sequence of tangent spaces $\{T_{x_k} X\}_{k \in \mathbb{N}}$ converges to $\tau \in \mathbb{G}_n^{\dim X}$, and the sequence of vectors $\frac{x_k - y_k}{\|x_k - y_k\|}$ converges to a unit vector v , one has $v \in \tau$.

Kuo's ratio test (r)— $\delta(T_{\pi_Y(x)} Y, T_x X) \ll \frac{\|x - \pi_Y(x)\|}{\|x - z\|}$, where $x \in X$ and x is converging to z .

Verdier condition (w)— there exist a neighborhood U_z of z in \mathbb{R}^n and a constant $C > 0$ such that

$$\delta(T_y Y, T_x X) \leq C \|x - y\|, \quad \forall x \in U_z \cap X, \forall y \in U_z \cap Y.$$

Here, π_Y denotes the locally orthogonal projection onto Y and

$$\delta(M, N) := \begin{cases} \sup_{x \in M, \|x\|=1} \|x - P_N(x)\|, & \text{if } M \not\equiv 0 \\ 0, & \text{if } M \equiv 0 \end{cases}$$

for M, N are vector subspaces of \mathbb{R}^n , where P_N is the orthogonal projection from \mathbb{R}^n onto N .

Remark 2.2. In o-minimal setting, we have $(w) \Rightarrow (r) \Rightarrow (b)$. Moreover, if \mathcal{D} is polynomially bounded and $\dim Y = 1$ then $(r) \Leftrightarrow (b)$ (see [12], [10]).

3. PRELIMINARY RESULTS OF \mathcal{D} -SETS

Let A be a \mathcal{D} -subset of \mathbb{R}^n (consider A as a family of \mathcal{D} -subsets of \mathbb{R}^{n-k} parameterized by \mathbb{R}^k). Let $U \subset \mathbb{R}^k$. We denote by $A|_U := \{x = (q, t) \in \mathbb{R}^{n-k} \times \mathbb{R}^k : x \in A, t \in U\}$. Let $t \in \mathbb{R}^k$. The set $A_t := \{q \in \mathbb{R}^{n-k} : (q, t) \in A\}$ is called the **fibre of A at the point t** . Let $\varepsilon > 0$. The **neighborhood of A of radius ε** is defined as follows

$$\mathcal{N}(A, \varepsilon) := \{x \in \mathbb{R}^n : d(x, A) \leq \varepsilon\}, \quad (3.1)$$

where d denotes the Euclidean distance in \mathbb{R}^n . Assume that $\dim A = l$. For $r > 0$, we define

$$\psi(A, r) := \mathcal{H}^l \left(A \cap \mathbf{B}_{(0,r)}^n \right). \quad (3.2)$$

Proposition 3.1. *Let $A \subset \mathbb{R}^n \times \mathbb{R}^m$ be a \mathcal{D} -set. Consider A as a family of \mathcal{D} -sets in of dimensions at most l in \mathbb{R}^n parameterized by \mathbb{R}^m . Then, there exists a constant $C > 0$ such that for any $r > 0$ and any $t \in \mathbb{R}^m$, we have*

- (i) $\psi(A_t, r) \leq Cr^l$.
- (ii) If $l < n$, $\forall \varepsilon > 0$, then

$$\psi(\mathcal{N}(A_t, \varepsilon), r) \leq Cr^{n-1}\varepsilon.$$

Proof. We follow closely the proof of Propositions 3.06 and 3.07 in [13].

(i)— In the case $l = n$, the set $A_t \cap \mathbf{B}_{(0,r)}^n$ being included in the ball $\mathbf{B}_{(0,r)}^n$ for all $t \in \mathbb{R}^m$, the result is obvious (the constant C is $\mathcal{H}^l(\mathbf{B}_{(0,1)}^l)$). If $l < n$, by removing a \mathcal{D} -subset of dimension less than l , we can consider A_t as a finitely disjoint union of graphs of Lipschitz mappings after a possible change of coordinates (the number of these graphs are bounded by a constant independent of t , see [6], Proposition 1.4). The volume of such a graph is equivalent to the volume of its image under the projection onto \mathbb{R}^l . The conclusion then follows from the case $l = n$.

(ii)— We denote by

$$A_t(\alpha) := \{x \in \mathbb{R}^n : d(x, A_t) = \alpha\}.$$

Since $\dim A_t < n$, $A_t(\alpha)$ is a \mathcal{D} -set of dimension $n-1$. By the case (i), $\psi(A_t(\alpha), r) \leq Cr^{n-1}$. Then,

$$\begin{aligned} \psi(\mathcal{N}(A_t, \varepsilon), r) &= \int_{\mathcal{N}(A_t, \varepsilon) \cap \mathbf{B}_{(0,r)}^n} d\mathcal{H}^n \\ &\leq \int_0^\varepsilon \psi(A_t(\alpha), r) d\mathcal{H}^1(\alpha) \\ &\leq Cr^{n-1}\varepsilon. \end{aligned}$$

□

Lemma 3.2. *Let A be a closed \mathcal{D} -subset of \mathbb{R}^n containing the origin. Suppose Σ is a Whitney stratification of A and $\{0\} \in \Sigma$. Then, there exists an $r_0 > 0$ such that for every $0 < r < r' \leq r_0$, there is a deformation retract from $A \cap \mathbf{B}_{(0,r')}^n$ onto $A \cap \mathbf{B}_{(0,r)}^n$ which preserves strata of the stratification Σ , i.e. there is a continuous mapping*

$$F : A \cap \mathbf{B}_{(0,r')}^n \times [0, 1] \rightarrow A \cap \mathbf{B}_{(0,r')}^n$$

such that $F(x, 0) = x$, $F(x, 1) \in A \cap \mathbf{B}_{(0,r)}^n$, $F(x, 1)|_{x \in A \cap \mathbf{B}_{(0,r)}^n} = x$ and $F|_{S \times [0,1]} \subset S$, $\forall S \in \Sigma$.

Moreover,

$$\|F(x, t) - x\| \leq 2t|r' - r|. \quad (3.3)$$

Proof. Let $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$, $x \mapsto \|x\|$ the distance function to the origin. Choose r_0 such that for every $0 < r \leq r_0$, $\mathbf{S}_{(0,r)}^{n-1} \cap S$, $\forall S \in \Sigma$. The collection $\Sigma' := \{\mathbf{B}_{(0,r_0)}^n \cap S, S \in \Sigma\}$ is then a Whitney stratification of $A \cap \mathbf{B}_{(0,r_0)}^n$. Moreover, the restriction of ρ to each stratum of Σ' is submersive.

For $x \in S$, $S \in \Sigma'$, set $v(x) := P_x(\partial_x \rho)$ where $P_x : \mathbb{R}^n \rightarrow T_x S$ is the orthogonal projection from \mathbb{R}^n onto the tangent space of S at x . Since $\rho|_S$ is a submersion, $v(x) \neq 0$, $\forall x \neq 0$.

Given an $\varepsilon > 0$, because Σ' is a Whitney stratification, shrinking r_0 if necessary, we can assume that $\forall x \in A \cap \mathbf{B}_{(0,r_0)}^n$, $\|v(x) - \partial_x \rho\| \leq \varepsilon$.

Consider Σ' as a filtration of skeletons $A \cap \mathbf{B}_{(0,r_0)}^n = S^d \supset S^{d-1} \supset \dots \supset \{0\}$, where $d = \dim A \cap \mathbf{B}_{(0,r_0)}^n$. Notice that, generally, v is not continuous on S^d even though its restriction to each stratum in S^d is continuous. We first construct a continuous stratified vector field on S^d , say w , by induction on skeletons so that

$$\|w(x) - \partial_x \rho\| \leq c\varepsilon$$

where c stands for some constant.

For $d = 0$, we take $w = \mu = 0$. Otherwise, write $S^d = \mathring{S}^d \cup S^{d-1}$ where \mathring{S}^d is the union of the strata of dimension d in S^d . By the inductive hypothesis, μ is a continuous stratified vector field on S^{d-1} and $\|\mu(x) - \partial_x \rho\| \leq c\varepsilon$. By a result of du Plessis [5], μ can be extended to a continuous stratified vector field on S^d . We will use the same notation μ for this extension. Since μ and $\partial_x \rho$ both are continuous on S^d , for each point $y \in S^{d-1}$ we can choose a neighborhood U_y in \mathbb{R}^n such that for any $x \in S^d \cap U_y$, we have $\|\mu(x) - \mu(y)\| \leq \varepsilon$ and $\|\partial_x \rho - \partial_y \rho\| \leq \varepsilon$. The union $\bigcup_{y \in S^{d-1}} U_y$ is an open neighborhood of S^{d-1} in \mathbb{R}^n . Define $T := \bigcup_{y \in S^{d-1}} (U_y \cap \mathring{S}^d)$, and call it an open neighborhood of S^{d-1} in \mathring{S}^d . Let T' be another open neighborhood of S^{d-1} in \mathring{S}^d such that the closure of T' in \mathring{S}^d is contained in T . Then we can choose a smooth partition $\{g_1, g_2\}$ of unity whose support refines $\{\mathring{S}^d \setminus T', T\}$, and define

$$w(x) = \begin{cases} \mu, & x \in S^{d-1} \\ g_1 v + g_2 \mu, & x \in \mathring{S}^d \end{cases}$$

It is clear that w is a continuous stratified vector field. Now we show that $\|w(x) - \partial_x \rho\| \leq c\varepsilon$. It suffices to check that the formula holds for every $x \in T$ since otherwise $w(x) = \mu(x)$ if $x \in S^{d-1}$ and $w(x) = v(x)$ if $x \in \mathring{S}^d \setminus T$ and obviously the formula holds.

Suppose $x \in T$. By the construction of T , there is $y \in S^{d-1}$ such that $\|\mu(x) - \mu(y)\| \leq \varepsilon$ and $\|\partial_x \rho - \partial_y \rho\| \leq \varepsilon$. Hence,

$$\begin{aligned} \|w(x) - \partial_x \rho\| &= \|g_1(v(x) - \partial_x \rho) + g_2(\mu(x) - \partial_x \rho)\| \leq \varepsilon + \|\mu(x) - \partial_x \rho\| \\ &\leq \varepsilon + \|\mu(x) - \mu(y)\| + \|\mu(y) - \partial_y \rho\| + \|\partial_y \rho - \partial_x \rho\| \\ &\leq \varepsilon + \varepsilon + c\varepsilon + \varepsilon = (3 + c)\varepsilon. \end{aligned}$$

Since w and $\partial_x \rho$ are continuous and $\|w(x) - \partial_x \rho\| \leq c\varepsilon$, $\|\partial_x \rho\| = 1$,

$$\xi(x) = \begin{cases} 0, & \text{if } x = 0 \\ \frac{-w(x)}{\langle \partial_x \rho, w(x) \rangle}, & \text{otherwise} \end{cases}$$

is well-defined and continuous on $S^d \setminus \{0\}$. We can, moreover, choose ε small enough so that $\|\xi(x)\| < 2$.

Write $\Phi(x, t)$ as the flow generated by the vector field ξ . Since $\rho_*(\xi(x)) = -1$, if $x \in X \cap \mathbf{B}_{(0,r)}^n$ then $\Phi(x, s) \in X \cap \mathbf{B}_{(0,r-s)}^n$ for all $r \leq r_0$. The map defined as follows

$$F(x, t) = \begin{cases} \Phi(x, t(\|x\| - r)), & \text{if } x \in X \cap (\mathbf{B}_{(0,r')}^n - \mathbf{B}_{(0,r)}^n) \\ x, & \text{if } x \in X \cap \mathbf{B}_{(0,r)}^n \end{cases}$$

is actually the desired deformation retract. \square

Remark 3.3. As in the proof, we are able to require $|\xi| < C$ with any constant $C > 1$ by taking ε sufficiently small.

4. CONTINUITY OF LOCAL LIPSCHITZ KILLING CURVATURES

4.1. Lipschitz Killing curvatures of \mathcal{D} sets. Let A be a compact \mathcal{D} -subset of \mathbb{R}^n . By Uniform Bound on Fibers and Hardt's triviality theorem, there is an integer $N \geq 0$ such that for every $P \in \mathbb{G}_n^k$, $0 \leq k \leq n$, there is a partition of P into \mathcal{D} -sets

$$K_{k,j}^P(A) := \{x \in P : \chi(\pi_P^{-1}(x) \cap A) = j\},$$

$j \in \{-N, -N+1, \dots, N\}$. The formula (1.1), therefore, becomes

$$\Lambda_k(A) = c(n, k) \int_{P \in \mathbb{G}_n^k} \sum_{j=1}^N j \mathcal{H}^k(K_{k,j}^P(A)) dP. \quad (4.1)$$

Let $x \in A$. By the same argument as in the proof of Theorem 1.3 [3], we can show that the local Lipschitz Killing curvatures $\Lambda_k^{\text{loc}}(A, x)$ of A at x (see the formula (1.2)) are well defined. It easy to verify that $\Lambda_0^{\text{loc}}(A, x) = 1$ and $\Lambda_k^{\text{loc}}(A, x) = 0$ for every $k > \dim A$.

4.2. Whitney condition (b). In this section, we suppose that \mathcal{D} is a polynomially bounded o-minimal structure, A is a closed \mathcal{D} -subset of \mathbb{R}^n with a Whitney stratification Σ . Let Y be a stratum of Σ . We may assume locally that $Y = \{0\}^{n-k} \times \mathbb{R}^k$ where $k = \dim Y$ and $\Sigma = \{Y, X_1, \dots, X_m\}$ with $Y \subset \text{cl}(X_i) \setminus X_i$, $\forall i \in \{1, \dots, m\}$.

Lemma 4.1. *Suppose that $\gamma(-\varepsilon, \varepsilon)$ is a C^1 \mathcal{D} -curve in Y , $\gamma(0) = 0$. Then, there exist $\nu > 0$ and a germ of homeomorphism*

$$h : A|_{\gamma([0, \nu])} \rightarrow A_0 \times \gamma([0, \nu]), \quad h(q, t) = (h_t(q), 0, t),$$

and constants C_1, C_2 such that

$$\begin{aligned} \|h_t(q) - q\| &\leq C_1 \|t\|^{1-e} r \\ \|h_t^{-1}(q) - q\| &\leq C_2 \|t\|^{1-e} r, \end{aligned}$$

$\forall (0, t) \in \gamma([0, \nu])$, $\forall q \in \mathbf{B}_{(0,r)}^{n-k}$ for which the above mappings are well-defined with r sufficiently close to zero.

Proof. We may write $\Sigma = \{Y = X_0, X_1, \dots, X_m\}$. Let $Y' := \gamma(-\varepsilon, \varepsilon)$. Denote by $\Sigma' := \{Y', X_1|_{Y'}, \dots, X_m|_{Y'}\}$. Then, Σ' is a Whitney stratification of $A|_{Y'}$ (consider in a neighborhood of Y' in \mathbb{R}^n).

Since $\dim Y' = 1$ and Σ' is a Whitney stratification, Σ' satisfies the condition (r) along Y' (see Remark 2.2). As proven in [11], there exist $0 \leq e < 1$ and a constant $C > 0$ such that

$$\delta(T_{\pi(x)}Y', T_x(X_i|_{Y'}))\|\pi(x)\|^e \leq C\|x - \pi(x)\|, \quad (4.2)$$

where $x \in X_i|_{Y'} \cap U$ with U is a small neighborhood in \mathbb{R}^n of the origin, $\pi : \mathbb{R}^n \rightarrow \{0\}^{n-k} \times \mathbb{R}^k$, $\pi(q, t) = (0, t)$.

Let $\mu(x) := -\gamma'(x)/\|\gamma'(x)\|$. Set $w(x) := P_x(\mu(\pi(x))) \in T_x(X_i|_{Y'})$, where P_x denotes the orthogonal projection onto the tangent space $T_x(X_i|_{Y'})$, $x \in X_i|_{Y'} \cap U$. Notice that $w(x) = \mu(x)$ for every $x \in Y'$. It follows from (4.2) that

$$\|w(x) - w(\pi(x))\|\|\pi(x)\|^e \leq C\|x - \pi(x)\|. \quad (4.3)$$

The condition Whitney (a) ensures that w and $\pi_*(w(x))$ are bounded below away from 0, where π_* denotes the tangent map of π . Thus, the vector field

$$v(x) := \frac{w(x)}{\|\pi_*(w(x))\|},$$

is well defined, satisfying (4.3) and $\pi_*(v(x)) = v(\pi(x)) = \mu(\pi(x))$.

Denote by Φ the flow generated by the vector field v . Let $x = (q, t) \in A|_{Y'} \cap U$, $(0, t) \in Y'$, and denote by $\sigma(t)$ the length of the arc γ from the origin to the point $(0, t)$. By shrinking U if necessary, we may assume that $\sigma(t) \sim \|t\|$ for all $(0, t) \in Y' \cap U$.

For $0 < s \leq \sigma(t)$, let $f(s) := \|\Phi_x(s) - \pi(\Phi_x(s))\|$. We have

$$f'(s) = \frac{\langle \Phi'_x(s) - \pi(\Phi'_x(s)), \Phi_x(s) - \pi(\Phi_x(s)) \rangle}{\|\Phi_x(s) - \pi(\Phi_x(s))\|} \leq \|\Phi'_x(s) - \pi(\Phi'_x(s))\|.$$

Here

$$\|\Phi'_x(s) - \pi(\Phi'_x(s))\| = \|v(\Phi_x(s)) - v(\pi(\Phi_x(s)))\|.$$

Associating with (4.3),

$$|f'/f| \leq C\|\pi(\Phi_x(s))\|^{-e}.$$

Since $\pi(\Phi_x(s)) = \Phi_{\pi(x)}(s)$ and $\|\Phi_{\pi(x)}(s)\| \sim s$, $|f'/f| \lesssim s^{-e}$. By integrating with respect to s over $[0, s]$ (note that $f(0) = \|q\|$) we get

$$\exp\left(\frac{-s^{1-e}}{1-e}\right)\|q\| \lesssim f(s) \lesssim \exp\left(\frac{s^{1-e}}{1-e}\right)\|q\|.$$

or, equivalently,

$$f(s) \sim \|q\|.$$

Write

$$\Phi_x(\sigma(t)) = (\Phi_x^1(\sigma(t)), \Phi_x^2(\sigma(t))) \in \mathbb{R}^{n-k} \times \mathbb{R}^k.$$

Then,

$$\begin{aligned}
\|\Phi_x^1(\sigma(t)) - q\| &= \left\| \int_0^{\sigma(t)} \frac{d}{ds}(\Phi_x(s) - \pi(\Phi_x(s)))ds \right\| = \left\| \int_0^{\sigma(t)} (\Phi'_x(s) - \pi(\Phi'_x(s)))ds \right\| \\
&= \left\| \int_0^{\sigma(t)} (v(\Phi_x(s)) - v(\pi(\Phi_x(s))))ds \right\| \\
&\leq \int_0^{\sigma(t)} \frac{\|\Phi_x(s) - \pi(\Phi_x(s))\|}{\|\pi(\Phi_x(s))\|^e} ds \lesssim \int_0^{\sigma(t)} f(s)s^{-e} ds \\
&\lesssim \|q\| \int_0^{\sigma(t)} s^{-e} ds \lesssim \|q\|\sigma(t)^{1-e} \lesssim \|q\|\|t\|^{1-e}.
\end{aligned}$$

Hence, the desired homeomorphism is given by $h_t(q) = \Phi_{(q,t)}^1(\sigma(t))$. \square

The following proposition is the key result for the proof of the main theorem.

Proposition 4.2. *Fix $0 \leq l \leq n-k$. There exists $C > 0$ such that for every $\varepsilon > 0$, there exists a neighborhood U_ε of 0 in Y such that $\forall(0, t) \in U_\varepsilon$, $\exists r_t > 0$, $\forall P \in \mathbb{G}_{n-k}^l$ and for every $0 < r \leq r_t$, there is a \mathcal{D} -subset $\Delta(P, \varepsilon, r, t)$ of P with*

$$\psi((\Delta(P, \varepsilon, r, t)), r) \leq C\varepsilon r^l$$

such that for any $x \in (\mathbf{B}_{(0,r)}^{n-k} \cap P) \setminus \Delta(P, \varepsilon, r, t)$,

$$\chi\left(\pi_P^{-1}(x) \cap A_t \cap \mathbf{B}_{(0,r)}^{n-k}\right) = \chi\left(\pi_P^{-1}(x) \cap A_0 \cap \mathbf{B}_{(0,r)}^{n-k}\right), \quad (4.4)$$

where π_P is the orthogonal projection from \mathbb{R}^{n-k} onto P .

Proof. Since Σ is a Whitney stratification, for each $t \in Y$ there is a $r_t > 0$ such that $\forall r \leq r_t$, the collection $\{S_t \cap \mathbf{B}_{(0,r)}^{n-k}, S_t \cap \mathbf{S}_{(0,r)}^{n-k-1}\}_{S \in \Sigma}$ forms a Whitney stratification of $A_t \cap \mathbf{B}_{(0,r)}^{n-k}$, denoted by \mathcal{S}_t^r . By Lemma 3.2, shrinking r_t if necessary, we may assume that for $0 < r < r' < r_t$, there is a deformation retract

$$F_t^{r,r'} : A_t \cap \mathbf{B}_{(0,r')}^{n-k} \times [0, 1] \rightarrow A_t \cap \mathbf{B}_{(0,r')}^{n-k}$$

from $A_t \cap \mathbf{B}_{(0,r')}^{n-k}$ onto $A_t \cap \mathbf{B}_{(0,r)}^{n-k}$ preserving the strata of $\mathcal{S}_t^{r_t}$ and

$$\|F_t^{r,r'}(q, s) - F_t^{r,r'}(q, 0)\| \leq 2s|r' - r|.$$

For $P \in \mathbb{G}_{n-k}^l$, consider the restriction of π_P to $A_0 \cap \mathbf{B}_{(0,r+2\varepsilon r)}^{n-k} \cup A_t \cap \mathbf{B}_{(0,r+3\varepsilon r)}^{n-k}$. By Hardt's triviality theorem, there exists a partition of $P \cap \mathbf{B}_{(0,r+3\varepsilon r)}^{n-k}$ into finitely many \mathcal{D} -sets such that $A_0 \cap \mathbf{B}_{(0,r+2\varepsilon r)}^{n-k} \cup A_t \cap \mathbf{B}_{(0,r+3\varepsilon r)}^{n-k}$ is definably trivial along elements of the partition (with respect to the projection map π_P) and the trivialization over these elements is compatible with all strata of $\mathcal{S}_0^r, \mathcal{S}_0^{r+2\varepsilon r}, \mathcal{S}_t^r, \mathcal{S}_t^{r+\varepsilon r}, \mathcal{S}_t^{r+3\varepsilon r}$.

Denote by $\Delta_{P,r,t}^1, \dots, \Delta_{P,r,t}^\nu$ the elements of dimension l of the partition and $\partial\Delta_{P,r,t}^1, \dots, \partial\Delta_{P,r,t}^\nu$ their corresponding topological boundaries. Set

$$\Delta(P, \varepsilon, r, t) := \bigcup_{i=1}^\nu \mathcal{N}(\partial\Delta_{P,r,t}^i, 10\varepsilon r).$$

Clearly, $\Delta(P, \varepsilon, r, t)$ is a \mathcal{D} -subset of P of dimension less than l . By Proposition 3.1, we have

$$\psi(\Delta(P, \varepsilon, r, t), r) \leq C\varepsilon r^l$$

for some $C > 0$ (note that C is independent of (P, ε, r, t)).

Let $x \in P$. We define

$$A_t^r(x, \lambda) := A_t \cap \mathbf{B}_{(0,r)}^{n-k} \cap \pi_P^{-1}(\mathbf{B}_{(x,\lambda)}^{n-k} \cap P).$$

Claim: For $x \in (\mathbf{B}_{(0,r)}^{n-k} \cap P) \setminus \Delta(P, \varepsilon, r, t)$, the homomorphisms of homology groups induced by the following inclusion maps

$$U_1 := A_0^r(x, \varepsilon r) \hookrightarrow U_2 := A_0^{r+2\varepsilon r}(x, 3\varepsilon r) \quad (\text{I})$$

$$W_1 := A_t^{r+\varepsilon r}(x, 2\varepsilon r) \hookrightarrow W_2 := A_t^{r+3\varepsilon r}(x, 4\varepsilon r) \quad (\text{II})$$

$$W_3 := A_t^r(x, \varepsilon r) \hookrightarrow W_2 := A_t^{r+3\varepsilon r}(x, 4\varepsilon r) \quad (\text{III})$$

are isomorphisms.

We will give the proof of (I) (using the same arguments we can get the proofs of (II) and (III)).

Let $x \in \mathbf{B}_{(0,r)}^{n-k} \cap P \setminus \Delta(P, \varepsilon, r, t)$. There exists $j \in \{1, \dots, \nu\}$ such that $x \in \Delta_{P,r,t}^j$. By the definition of $\Delta_{P,r,t}^j$, $\mathbf{B}_{(x,10\varepsilon r)}^{n-k} \cap P \subset \Delta_{P,r,t}^j$. Since the trivialization of $A_0 \cap \mathbf{B}_{(0,r+2\varepsilon)}^{n-k} \cup A_t \cap \mathbf{B}_{(0,r+3\varepsilon r)}^{n-k}$ over $\Delta_{P,r,t}^j$ is compatible with all strata of \mathcal{S}_0^r and $\mathcal{S}_0^{r+2\varepsilon r}$, $\forall \varrho \in \{r, r+2\varepsilon r\}$ and $\forall \lambda, \lambda'$ such that $0 < \lambda < \lambda' \leq 10\varepsilon r$, there is a deformation retract, denoted by $\Psi_0^\varrho(x, \lambda, \lambda')$, from $A_0^\varrho(x, \lambda')$ onto $A_0^\varrho(x, \lambda)$ which preserves the strata of \mathcal{S}_0^ϱ .

Set

$$U_2' := A_0^{r+2\varepsilon r}(x, 7\varepsilon r)$$

and

$$V_1 := \{F_0^{r,r+2\varepsilon r}(q, s), q \in U_2, s \in [0, 1]\}$$

$$V_2 := \{\Psi_0^r(x, \varepsilon r, 7\varepsilon r)(q, s), q \in F_0^{r,r+2\varepsilon r}(U_2, 1), s \in [0, 1]\}$$

$$V := V_1 \cup V_2.$$

It is obvious that U_1 is a retract of V by the map defined as follows

$$G : V \times [0, 1] \rightarrow V, \quad G(q, s) = \begin{cases} F_0^{r,r+2\varepsilon r}(q, 2s), & q \in V_1, s \leq \frac{1}{2} \\ \Psi_0^r(x, \varepsilon r, 7\varepsilon r)(q, 2s-1), & q \in V_2, s > \frac{1}{2}. \end{cases}$$

Consider the following commutative diagram of homology groups induced by inclusion maps.

$$\begin{array}{ccccc} & & & & H_*(V) \\ & & & \nearrow \alpha'_* & \\ H_*(U_1) & \xrightarrow{\alpha_*} & H_*(U_2) & \xrightarrow{\mu_*} & \\ & \searrow \gamma_* & \downarrow \beta_* & \nearrow \beta'_* & \\ & & H_*(U_2') & & \end{array}$$

Since U_1 and U_2 are retracts of V and U'_2 respectively, the maps α'_* and β_* are isomorphisms. Hence, homomorphisms in the diagram above are isomorphisms. This establishes claim (I).

Now we are ready to prove the proposition.

Case 1: $\dim Y = 1$. First, we choose a neighborhood U_ε of 0 sufficiently small so that Lemma 4.1 holds, this means there are $0 \leq e < 1$, $c > 0$, for every $(0, t) \in U_\varepsilon$ there exist $r_t > 0$ and a homeomorphism $h_t : (A_t, 0) \rightarrow (A_0, 0)$ such that

$$\|h_t(q) - q\| \leq c\|t\|^{1-e}r, \quad \forall q \in A_t \cap \mathbf{B}_{(0,r)}^{n-k}, r \leq r_t.$$

Shrinking U_ε if necessary, we can assume that $c\|t\|^{1-e} < \varepsilon$, $\forall (0, t) \in U_\varepsilon$. This implies that

$$U_1 \subset h_t(W_1) \subset U_2 \subset h_t(W_2).$$

Consider the following commutative diagram induced by inclusion maps

$$\begin{array}{ccccc} H_*(U_1) & \xrightarrow{\iota_{1*}} & H_*(h_t(W_1)) & & \\ & \searrow u_* & \downarrow \iota_* & \searrow w_* & \\ & & H_*(U_2) & \xrightarrow{\iota_{2*}} & H_*(h_t(W_2)) \end{array}$$

By the claim, we have that u_* and w_* are isomorphisms. Since the diagram commutes, it is easy to show that ι_* , ι_{1*} , ι_{2*} are isomorphisms. Finally,

$$\begin{aligned} H_*\left(\pi_P^{-1}(x) \cap A_t \cap \mathbf{B}_{(0,r)}^{n-k}\right) &= H_*(U_1) = H_*(h_t(W_1)) = H_*(W_2) \\ &= H_*(W_3) = H_*\left(\pi_P^{-1}(x) \cap A_0 \cap \mathbf{B}_{(0,r)}^{n-k}\right). \end{aligned}$$

Case 2: $\dim Y > 1$. We define

$$\begin{aligned} \Omega := & \left\{ (0, t) \in Y : \forall \varepsilon > 0, \exists \sigma > 0, \forall r \in (0, \sigma), \forall P \in \mathbb{G}_{n-k}^l, \right. \\ & \left[\forall x \in \mathbf{B}_{(0,r)}^{n-k} \cap P \setminus \Delta(P, \varepsilon, r, t) \right. \\ & \left. \Rightarrow \left[\chi\left(\pi_P^{-1}(x) \cap A_t \cap \mathbf{B}_{(0,r)}^{n-k}\right) = \chi\left(\pi_P^{-1}(x) \cap A_0 \cap \mathbf{B}_{(0,r)}^{n-k}\right) \right] \right\}. \end{aligned}$$

Since $\chi(P, \varepsilon, x, t, r) := \chi\left(\pi_P^{-1}(x) \cap A_t \cap \mathbf{B}_{(0,r)}^{n-k}\right)$ is a \mathcal{D} -function, Ω is a \mathcal{D} -set. It suffices to prove that the set Ω contains a neighborhood of 0. This fact actually follows directly from Curve Selection and Case 1. \square

Theorem 4.3 (Main Theorem). *The local Lipschitz Killing curvatures of A are continuous along the strata of Σ .*

Proof. We will work with the following family of \mathcal{D} -sets.

$$\mathcal{A} := \{(x, u, t) \in \mathbb{R}^{n-k} \times \mathbb{R}^k \times \mathbb{R}^k : (x, u + t) \in A\}.$$

It is obvious that \mathcal{A} is the image of $A \times \mathbb{R}^k$ under the linear isomorphism

$$\varphi : A \times \mathbb{R}^k \rightarrow \mathcal{A}, \quad (q, u, t) \mapsto (q, u - t, t).$$

The germ of A at $(0, t)$ can be viewed as the germ of \mathcal{A}_t at $(0, 0)$. Therefore, the continuity of $\Lambda_l^{\text{loc}}(A, t)$ along Y is equivalent to the continuity of $\Lambda_l^{\text{loc}}(\mathcal{A}_t, 0)$ along $\{0\}^n \times \mathbb{R}^k$.

Set

$$\Delta := \{(u, t) \in \mathbb{R}^k \times \mathbb{R}^k : u = t\}.$$

Since $\Sigma = \{Y = \{0\}^{n-k} \times \mathbb{R}^k, X_1, \dots, X_m\}$ is a Whitney stratification of A , the collection $\{\{0\}^{n-k} \times \mathbb{R}^k \times \mathbb{R}^k, X_1 \times \mathbb{R}^k, \dots, X_m \times \mathbb{R}^k\}$ is a Whitney stratification of $A \times \mathbb{R}^k$, and so is $\Sigma' := \{\{0\}^{n-k} \times \Delta, \{0\}^{n-k} \times (\mathbb{R}^k \times \mathbb{R}^k \setminus \Delta), X_1 \times \mathbb{R}^k, \dots, X_m \times \mathbb{R}^k\}$. We denote by Σ'' the collection of images of all strata of Σ' under the map φ . Then, Σ'' is a Whitney stratification of \mathcal{A} containing $\varphi(\{0\}^{n-k} \times \Delta) = \{0\}^n \times \mathbb{R}^k$ as a stratum. It suffices to show that $\Lambda_l^{\text{loc}}(\mathcal{A}_t, 0)$ is continuous in t along this stratum.

Applying Proposition 4.2 to the stratified set (\mathcal{A}, Σ'') , with $0 \leq l \leq n$ fixed, there is $C > 0$, for every $\varepsilon > 0$, there is a neighborhood U_ε in $\{0\}^n \times \mathbb{R}^k$ of the origin such that for any $(0, t) \in U_\varepsilon$, there is a $r_t > 0$ such that for any $0 < r \leq r_t$, for every $P \in \mathbb{G}_n^l$, there exists a \mathcal{D} -subset $\Delta(P, \varepsilon, t, r)$ of P with

$$\psi(\Delta(P, \varepsilon, t, r), r) \leq C\varepsilon r^l$$

such that for any $x \in (\mathbf{B}_{(0,r)}^n \cap P) \setminus \Delta(P, \varepsilon, t, r)$:

$$\chi(\pi_P^{-1}(x) \cap \mathcal{A}_t \cap \mathbf{B}_{(0,r)}^n) = \chi(\pi_P^{-1}(x) \cap \mathcal{A}_0 \cap \mathbf{B}_{(0,r)}^n).$$

It follows from (4.1) that for any j and for any $P \in \mathbb{G}_n^l$, we have

$$\psi\left(K_{l,j}^P(\mathcal{A}_0 \cap \mathbf{B}_{(0,r)}^n) \setminus K_{l,j}^P(\mathcal{A}_t \cap \mathbf{B}_{(0,r)}^n), r\right) \leq C\varepsilon r^l$$

and,

$$\psi\left(K_{l,j}^P(\mathcal{A}_t \cap \mathbf{B}_{(0,r)}^n) \setminus K_{l,j}^P(\mathcal{A}_0 \cap \mathbf{B}_{(0,r)}^n), r\right) \leq C\varepsilon r^l.$$

Thus, we get

$$|\psi\left(K_{l,j}^P(\mathcal{A}_0 \cap \mathbf{B}_{(0,r)}^n)\right) - \psi\left(K_{l,j}^P(\mathcal{A}_t \cap \mathbf{B}_{(0,r)}^n), r\right)| \leq C\varepsilon r^l.$$

By formula (1.2),

$$|\Lambda_l\left(\mathcal{A}_0 \cap \mathbf{B}_{(0,r)}^n\right) - \Lambda_l\left(\mathcal{A}_t \cap \mathbf{B}_{(0,r)}^n\right)| \leq C\varepsilon r^l.$$

Dividing by r^l , we obtain

$$|\Lambda_l^{\text{loc}}(\mathcal{A}_0, 0) - \Lambda_l^{\text{loc}}(\mathcal{A}_t, 0)| \leq C\varepsilon.$$

The theorem is proved. \square

4.3. Kuo-Verdier condition (w). In this section, we assume that \mathcal{D} is an arbitrary o-minimal structure and Σ is a (w)-regular stratification of A . The other hypotheses remain as in Section 4.2. We first establish results of the same types as Lemma 4.1 and Proposition 4.2.

Lemma 4.4. *There exist a neighborhood U of 0 in Y and constants $C_1, C_2 > 0$ such that for every $(0, t')$ in U , there is a germ of homeomorphism*

$$h : A|_U \rightarrow A_{t'} \times U, \quad h_t(q, t) = (h_t(q), 0, t)$$

satisfying

$$\begin{aligned}\|h_t(q) - q\| &\leq C_1 \|t - t'\| r \\ \|h_t^{-1}(q) - q\| &\leq C_2 \|t - t'\| r\end{aligned}$$

where $q \in \mathbf{B}_{(0,r)}^{n-k}$ for which the mapping h_t is well-defined, r is sufficiently small.

Proof. The argument here is classical which can be found in [9] [16]. Consider the coordinate vector fields $\partial_1, \dots, \partial_k$ in $\{0\}^{n-k} \times \mathbb{R}^k$. There are corresponding rugose vector fields $\tilde{\partial}_1, \dots, \tilde{\partial}_k$ on A (consider in a neighborhood of 0) such that

$$\pi_*(\tilde{\partial}_\alpha) = \partial_\alpha, \quad \alpha \in \{1, \dots, k\},$$

where $\pi : \mathbb{R}^{n-k} \times \mathbb{R}^k \rightarrow \{0\}^{n-k} \times \mathbb{R}^k$, $\pi(q, t) = (0, t)$ (see [16], Proposition 4.6).

By the rugosity of $\tilde{\partial}_\alpha$, there is a neighborhood V of 0 such that for all $x \in V \cap A$,

$$\|\tilde{\partial}_\alpha(x) - \tilde{\partial}_\alpha(\pi(x))\| \leq C \|x - \pi(x)\|. \quad (4.5)$$

Denote by Φ_α the flow generated by the vector field $\tilde{\partial}_\alpha$. We write

$$\Phi_\alpha(x, s) = (\Phi_\alpha^1(x, s), \Phi_\alpha^2(x, s)) \in \mathbb{R}^{n-k} \times \mathbb{R}^k.$$

Since $\tilde{\partial}_\alpha$ satisfies the formula (4.5), by the same computation as in the proof of Lemma 4.1 we have

$$\|\Phi_\alpha^1(x, s) - q\| \leq C |s| \|q\|, \quad x = (q, t).$$

Define

$$h(x) = (\Phi_1(\dots(\Phi_{k-1}(\Phi_k(x, t'_k - t_k), t'_{k-1} - t_{k-1}), \dots), t'_1 - t_1), t),$$

where $t = (t_1, \dots, t_k)$ and $t' = (t'_1, \dots, t'_k)$. It is easy to check that h is the desired homeomorphism. \square

Proposition 4.5. *Fix $0 \leq l \leq n - k$. There exist a constant $C > 0$ and a neighborhood U of 0 in Y such that for every $(0, t)$ and $(0, t')$ in U , $\exists r_{t,t'} > 0$, for every $P \in \mathbb{G}_{n-k}^l$ and $0 < r \leq r_{t,t'}$, there is a \mathcal{D} -subset $\Delta(P, r, t, t')$ of P with*

$$\psi((\Delta(P, r, t, t')), r) \leq C \|t - t'\| r^l$$

such that for any $x \in (\mathbf{B}_{(0,r)}^{n-k} \cap P) \setminus \Delta(P, r, t, t')$,

$$\chi(\pi_P^{-1}(x) \cap A_t \cap \mathbf{B}_{(0,r)}^{n-k}) = \chi(\pi_P^{-1}(x) \cap A_{t'} \cap \mathbf{B}_{(0,r)}^{n-k}).$$

Proof. Choose a neighborhood U of 0 in Y sufficiently small so that Lemma 4.4 holds, i.e., there exists $c > 0$, for every $(0, t)$ and $(0, t')$ in U , there is a homeomorphism $h_{t,t'} : A_t \rightarrow A_{t'}$ and $r_{t,t'} > 0$ such that

$$\|h_{t,t'}(q) - q\| \leq c \|t - t'\| r, \quad \forall q \in A_t \cap \mathbf{B}_{(0,r)}^{n-k}, \forall r \leq r_{t,t'}.$$

Applying the same arguments as in the proof of Proposition 4.2 (Case 1) (just replace ε with $\|t - t'\|$ and consider t' as the origin) we obtain the desired result. \square

By using Proposition 4.5 and the same arguments as in the proof of Theorem 4.3 we get:

Theorem 4.6. *The local Lipschitz Killing curvatures of A are locally Lipschitz along the strata of Σ .*

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